

Pseudoscalar Glueball, η' -meson and its Excitation in the Chiral Effective Lagrangian

M. L. Nekrasov *

Institute for High Energy Physics, 142284 Protvino, Russia

Abstract

A generalization of the chiral effective lagrangian of order p^2 is proposed which involves the η' -meson, its excitation, and the pseudoscalar (PS) glueball. Model-independent constraints are found for the contributions to the lagrangian of the above singlet states. Those allow one to independently identify the nature of these singlet states in the framework of the approach. The mixing among the iso-singlet states (including η^8 -state) is analysed, and the hierarchy of the mixing angles is described which is defined by the chiral and large- N_c expansions. The recent PCAC results are reproduced, which are related to the problem of the renormalization-group invariant description of the η' and the PS glueball, and a further analysis of this problem is performed.

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*E-mail: nekrasov@mx.ihep.su

1 Introduction

The problem of the description of glueballs in QCD is longstanding but still unsolved. In spite of the consensus which has been recently achieved in the gluonic-lattice calculations [1], there remains a serious problem of taking into account quark contributions. The problem seems to be most intricate in pseudoscalar (PS) channel, where there is the η' -meson which, being quarkic in its origin, contains a gluonic contribution that significantly affects its observable properties [2, 3]. Consequently, one may expect that quarks are quite significant in the formation of the PS glueball.

In fact, the situation is more complicated than it is usually assumed, since in the exact QCD the quark-gluon mixing may depend on the ultraviolet renormalization scale [4]. As a rule, this phenomenon is ignored. However, in some cases that leads to disastrous effects. Indeed, it has been found in [5, 6] that the straightforward generalization of the PCAC formula for $\pi \rightarrow \gamma\gamma$ to $\eta' \rightarrow \gamma\gamma$ is inconsistent with the renormalization group (RG) and therefore incorrect in principle. To obtain a correct formula one has to consider a set of composite operators which mix under RG [5]. A similar investigation was performed which involved both the η' -meson and the PS glueball [7], and there the RG invariant composite operators (interpolating fields) were obtained that generate separately each state. However, that investigation was ill-fitted for the description of the hadronic decays of the η' and the PS glueball. Therefore, the problem needs to be re-analysed in a more sophisticated fashion.

A systematic approach is based on the chiral effective lagrangian [8]. When it is considered in the framework of the chiral perturbation theory [9, 10] it allows one to describe consistently the low-energy interactions of the lightest pseudoscalar states π, K, η , and their interactions with heavier states [11]. The mixing phenomenon, as well, should be tractable in the framework of this approach. Concerning the mixing between the PS glueball and pseudoscalar quarkic states, the latter states might be, first of all, the η' and its excitations (including the radial ones and hybrids). How many states are needed depends on how heavy the PS glueball is. If its mass lies in the E/ι range (1.4–1.5 GeV) then, most probably, the η' and its first radial excitation would be enough to describe the mixing. If the PS glueball is heavier then apparently more states would be needed.

Of course, with singlet states, the chiral symmetry is no longer sufficient to constrain their contributions to the chiral effective lagrangian (except the case of η' which is conditioned by its $U(1)_A$ transformation property [2, 10]). Moreover, the chiral symmetry does not allow one to distinguish between different singlet states. Nevertheless, one may expect that if the dynamical nature of the singlet states is different then some extra dynamical (not symmetry) conditions would constrain the singlet-state contributions to the lagrangian. For instance, there is large- N_c expansion which leads to the condition that the gluonic and quarkic contributions must behave differently at large N_c [2, 12]. However, this condition cannot completely suppress any parameter of the chiral effective lagrangian. Therefore one needs to find some stronger conditions. The main task of the present paper is to try to find them. Another task is to reexamine the problem of the RG invariant description of the singlet states in the framework of the chiral effective lagrangian.

The structure of the paper is as follows. The next section collects the necessary

notation and discusses the conditions which follow from the chiral symmetry and RG symmetry in QCD. Section 3 discusses a generalization of the chiral effective lagrangian which involves singlet interpolating fields and satisfies the above symmetry conditions. In Section 4 we search for extra conditions which would constrain the singlet-state contributions. A consistency condition is found which additionally constrains the contributions of the η' and constrains as well the contributions of the other singlet PS states. A general necessary condition is proposed for glueballs. Moreover, a general physical criterion is proposed which allows one to distinguish between the ground and excited states in the framework of the approach. Combined together these results allow one to constrain the contributions of the ground-state PS glueball and an excitation state over η' . The structure of the iso-singlet states mixing is investigated in section 5. Section 6 discusses the problem of radiative decays of η and η' . Section 7 summarizes the results.

2 Symmetry conditions

A consistent way to introduce chiral effective lagrangian is through the generating functional [9, 10]. This method permits to establish a complete relation between the effective theory and the underlying theory (QCD). In the path-integral approach the generating functional may be written in the following equivalent representations:

$$\begin{aligned} e^{iW(V,A,S,P,\Theta)} &= \int \mathcal{D}[q, \bar{q}, G] e^{i \int d^4x \mathcal{L}_{QCD}(q, \bar{q}, G; V, A, S, P, \Theta)} \\ &= \int \mathcal{D}[U \dots] e^{i \int d^4x \mathcal{L}_{eff}(U \dots; V, A, S, P, \Theta)}. \end{aligned} \quad (1)$$

Here the first equality defines the generating functional in terms of QCD parameters with q , \bar{q} , and G being the fundamental fields of quarks and gluons. The quantities V, A, S, P, Θ are the sources of the composite operators which generate the states to be described. In our case those are the axial-vector and PS quark currents and their chiral partners. Besides, we have introduced the source for the gluon anomaly operator which generates the PS glueball. The notation is as follows:

$$\begin{aligned} \mathcal{L}_{QCD} &= \mathcal{L}_{QCD}^0 + \bar{q} \gamma_\mu (V_\mu + \gamma_5 A_\mu) q - \bar{q} (S + i\gamma_5 P) q + \Theta Q, \\ V &= \sum_{a=0,1,\dots,8} (\lambda^a/2) V^a, \quad \dots, \quad Q = \sqrt{2N_f} \frac{\alpha_s}{8\pi} \frac{1}{2} \epsilon_{\mu\nu\lambda\rho} G^{A\mu\nu} G^{A\lambda\rho}. \end{aligned} \quad (2)$$

Here λ^a are the flavor Gell-Mann matrices ($\lambda^0 = \sqrt{2/N_f} \mathbf{I}$, $N_f = 3$). With switched-off sources, $S = \text{diag}(m_u, m_d, m_s)$, $P = V = A = \Theta = 0$. The sources may also be regarded as the external fields which are provided with some certain transformation properties.

The second equality in (1) presents the generating functional of the effective theory which copies QCD in terms of the interpolating fields for observable states. Usually, all heavy states are considered to be integrated out. Then, the dynamical variables become only the interpolating fields for the octet of lightest PS states π, K, η . Since these states may be interpreted as Goldstone bosons that arise due to dynamical breaking of the

chiral symmetry $SU(3)_L \times SU(3)_R$, their interpolating fields may be collected in a special unitary 3×3 matrix U which under $SU(3)_L \times SU(3)_R$ transforms as

$$U \rightarrow \Omega_L U \Omega_R^\dagger. \quad (3)$$

In the exponential parameterization,

$$U = \exp \left(i \sum_{\alpha=1, \dots, 8} \lambda^\alpha \eta^\alpha / F \right) \quad (4)$$

with η^α being the interpolating fields and F being a dimensional parameter (when η^α are normalized canonically F is the universal octet decay constant). Under the ‘flavor-singlet’ chiral group $U(1)_L \times U(1)_R$ matrix U is invariant.

Remember, in QCD the full chiral group $U(3)_L \times U(3)_R$ acts on quarks and on the θ -vacuum. The latter property is developed in the appearance of the non-vanishing value of Θ when the $U(1)_A$ chiral rotation of quarks has been performed and the sources have been switched off:

$$\Theta|_{\text{switched-off}} = \omega_5^0. \quad (5)$$

Here $\omega_5^0 = (\omega_R^0 - \omega_L^0)/2$ is the parameter of $U(1)_A$ rotation. In spite of $U(3)_L \times U(3)_R$ transformation, lagrangian (2) may be made completely chiral invariant if one assumes the external fields to transform simultaneously by

$$\begin{aligned} L_\mu &\rightarrow \Omega_L L_\mu \Omega_L^\dagger + i \Omega_L \partial_\mu \Omega_L^\dagger, & L_\mu &= V_\mu - A_\mu \\ R_\mu &\rightarrow \Omega_R R_\mu \Omega_R^\dagger + i \Omega_R \partial_\mu \Omega_R^\dagger, & R_\mu &= V_\mu + A_\mu \\ M &\rightarrow \Omega_L M \Omega_R^\dagger, & M &= S + iP \\ \Theta &\rightarrow \Theta - \omega_5^0. \end{aligned} \quad (6)$$

Under this condition the transformation property of the generating functional (1) is only governed by the external chiral anomaly [13]. The requirement to reproduce this property in the effective theory leads to the condition that the effective lagrangian must be a sum of an invariant part and the Wess-Zumino-Witten term which is responsible for the external anomaly in the effective theory.

Usually, no other QCD symmetry is imposed while constructing \mathcal{L}_{eff} , because it is assumed that no other symmetry is able to constrain the effective theory. Nevertheless, as we pointed out in Introduction, the RG symmetry in QCD may lead to nontrivial consequences in the singlet channel. Therefore, this symmetry must be taken into consideration, as well. Earlier, that was not done since it was not well-known how to renormalize the underlying theory when the sources for composite operators are switched on. The problem was solved in [14] where it was stated that the renormalized action of the theory must be extended to include all possible terms formed from a single composite operator and an arbitrary number of sources and divergences. Furthermore, Ref. [14] showed that the unit operator $\mathbf{1}$ must be included into the basis of composite operators, since the renormalized action must include terms involving sources only. The generating

functional, then, becomes RG invariant if it is understood in terms of the renormalized sources [14]. In our case the latter ones are

$$\begin{aligned} S_R^a &= Z_m^{-1} S^a, & P_R^a &= Z_m^{-1} P^a, & (a = 0, 1, \dots, 8) \\ V_{\mu R}^\alpha &= V_\mu^\alpha, & A_{\mu R}^\alpha &= A_\mu^\alpha, & (\alpha = 1, \dots, 8) \\ V_{\mu R}^0 &= V_\mu^0, & A_{\mu R}^0 &= Z^{-1} A_\mu^0 + (1 - Z^{-1}) \partial_\mu \Theta, & \Theta_R = \Theta. \end{aligned} \quad (7)$$

Here Z_m and Z are the renormalization constants, index R shows the renormalized sources (external fields). Note, the inhomogeneous character of the renormalization of A_μ^0 means that there is renormalization-scale-dependent mixing between the gluon anomaly operator Q and divergence of the axial-vector singlet quark current $J_{\mu 5}^0 = \bar{q} \gamma_\mu \gamma_5 (\lambda^0/2) q$,

$$[Q]_R = Q - (1 - Z) \partial^\mu J_{\mu 5}^0, \quad [J_{\mu 5}^0]_R = Z J_{\mu 5}^0. \quad (8)$$

Besides (7), the auxiliary source of the unit operator must be renormalized, too [14]. However, since the corresponding formula involves only the terms of the *chiral* dimension 4 or higher, this renormalization rule (and the unit operator itself) may be disregarded when the effective theory is considered at order p^2 of the chiral expansion. So, the sole RG requirement which should be taken into consideration at order p^2 is the requirement that \mathcal{L}_{eff} must be invariant provided that the external fields are renormalized by (7).

3 Singlet interpolating fields in the chiral effective lagrangian

In what follows we will consider the effective theory in the framework of the chiral perturbation theory. That allows one to represent the effective lagrangian in the form of an expansion in derivatives of fields and quark masses. In case when only octet of the lightest PS states is involved the explicit form of the lagrangian is well-known [10]. At the leading order p^2 of the chiral expansion it is

$$\begin{aligned} \mathcal{L}_{eff} &= \frac{1}{4} F^2 \langle \nabla_\mu U \nabla^\mu U^\dagger \rangle + \frac{1}{2} B F^2 \langle M_\Theta U^\dagger + M_\Theta^\dagger U \rangle + \frac{1}{2} H \nabla_\mu \Theta \nabla^\mu \Theta \\ \nabla_\mu U &= \partial_\mu U - i \tilde{L}_\mu U + i U \tilde{R}_\mu, \quad \nabla_\mu \Theta = \partial_\mu \Theta - A_\mu^0, \quad M_\Theta = (S + iP) e^{i\lambda^0 \Theta}. \end{aligned} \quad (9)$$

Here $\langle \dots \rangle$ means the trace operation, the tildes mean that \tilde{L}_μ and \tilde{R}_μ are determined without the singlet fields L_μ^0 and R_μ^0 (i.e., $\tilde{L}_\mu^0 = \tilde{R}_\mu^0 = 0$). As a result, $\nabla_\mu U$ transforms like U , which is invariant under $U(1)_L \times U(1)_R$. The same property is also relevant for M_Θ . The singlet external fields L_μ^0 and R_μ^0 are both collected in the last term in (9), $\nabla_\mu \Theta$ is the chiral invariant derivative of Θ . The quantities F, B, H in (9) are the low-energy constants. Their physical significance may be established basing on the property that at the leading order p^2 the quantum loops do not contribute in the effective theory [10]. Therefore the generating functional is equal to the classical action,

$$L_{eff}(U; V, A, S, P, \Theta) = \int d^4x \mathcal{L}_{eff}(U; V, A, S, P, \Theta), \quad (10)$$

evaluated at the solution to the equations of motion for U . Owing to this property one can show that F is the decay constant of the axial-vector octet quark current, BF^2 is the quark condensate (with the opposite sign) in the chiral limit, H describes the low-energy asymptotic of the two-point Green function of the axial-vector singlet quark current.

It is easy to see that due to (3) and (6) the lagrangian (9) and generating functional (10) are both chiral invariant. Moreover, they are RG invariant as well. Really, owing to [14] and the above QCD description of the low-energy constants one can find that F is RG invariant, whereas constants B and H are renormalized as

$$B = Z_m^{-1} B_R, \quad H = Z^{-2} H_R. \quad (11)$$

Since the external fields are renormalized by (7), the RG invariance of lagrangian (9) is observed if matrix U is RG invariant. However, the latter property takes place due to the equations of motion. So, the RG invariance is really observed. Notice, since U and F are both RG invariant, the interpolating fields η^α ($\alpha = 1, \dots, 8$) must be RG invariant, too, as it should be in a consistent effective theory.

Now let allow for the presence of singlet states in the effective theory. A way to correspondingly generalize the chiral effective lagrangian is through the shift of the low-energy constants to invariant functions [10] which would describe the dependence on the singlet-state interpolating fields. Besides, one must add the necessary kinetic terms in order to describe the spectrum of the singlet states. Of course, in this way one cannot distinguish between different singlet states, and one must add some extra considerations to do that. This problem will be intensively discussed below. For the present, we only notice one special case of singlet state. It is the case of the lowest singlet quarkic state η^0 which interpolating field must transform by a shift under $U(1)_A$:

$$\eta^0 \rightarrow \eta^0 + F_0 \omega_5^0. \quad (12)$$

Condition (12) was first imposed in [2] in order to resolve in terms of the effective theory the paradox between the θ -dependence of the vacuum and the large- N_c behavior of QCD with massless quarks. The quantity F_0 is a dimensional parameter. In the limit of large N_c it must coincide with F [2], but its value remains unknown with N_c finite. Due to (12) and (6), the combination $\eta^0 + F_0 \Theta$ is completely chiral invariant. So, this very combination should be placed into the lagrangian, but not η^0 itself.

This important observation was made, and the corresponding generalization of the chiral effective lagrangian that involves η^0 was proposed [10]. However, the physical significance of F_0 , especially that in terms of QCD parameters, is not clarified yet. (Usually F_0 is equated to F without discussions.) This gap, of course, should necessarily be filled. Besides, there is another problem while involving η^0 . It is the necessity to observe the RG symmetry inspired by QCD. Previously, it was not taken into consideration in the singlet channel. As a result, the RG symmetry was lost. Consequently, for instance, the coupling of the singlet axial-vector current to η' was described wrong. This gap should be filled, too. Our nearest task is to solve these problems.

First, let us find the correct generalization of \mathcal{L}_{eff} which would involve η^0 and satisfy both the chiral and RG symmetry. Again, we will work up to order p^2 of the chiral expansion. In general case \mathcal{L}_{eff} may be represented in the form

$$\mathcal{L}_{eff} = \mathcal{L}^{(0)} + \mathcal{L}^{(kin)} + \mathcal{L}^{(mass)}, \quad (13)$$

where $\mathcal{L}^{(0)}$ involves η^0 only, without contributions of the octet interpolating fields. $\mathcal{L}^{(kin)}$ and $\mathcal{L}^{(mass)}$ in (13) involve both η^0 and the octet interpolating fields. It is convenient to take $\mathcal{L}^{(0)}$ in the form with the explicitly extracted quadratic terms:

$$\begin{aligned}\mathcal{L}^{(0)} = & \frac{1}{2}\nabla_\mu\eta^0\nabla^\mu\eta^0 - \frac{1}{2}M_0^2(\eta^0 + F_0\Theta)^2 \\ & + H_0\nabla_\mu\eta^0\nabla^\mu\Theta + \frac{1}{2}H\nabla_\mu\Theta\nabla^\mu\Theta + \mathcal{L}_{int}^{(0)}.\end{aligned}\quad (14)$$

Here

$$\nabla_\mu\eta^0 = \partial_\mu(\eta^0 + F_0\Theta) \quad (15)$$

is the chiral-invariant derivative of η^0 . Note, we have defined it differently as compared with [10] where $\nabla_\mu\eta^0$ was defined using A_μ^0 instead of $\partial_\mu\Theta$. Although from the point of view of the chiral symmetry both definitions are equivalent, our one (15) is preferable from the point of view of RG symmetry (see below). H_0 in (14) is a new dimensional low-energy constant (its physical significance will be discussed latter). M_0 is the mass of η^0 in the chiral limit (it may be related to the topological susceptibility of gluons in QCD without quarks [2]). The term $\mathcal{L}_{int}^{(0)}$ in (14) describes the η^0 self-interaction and its accompanying interaction with external fields. This term is irrelevant, however, when only the mixing and decays of η^0 are subjects of consideration. The ‘kinetic’ and the ‘mass’ terms in (13) are as follows

$$\mathcal{L}^{(kin)} = \frac{1}{4}F^2v_1\langle\nabla_\mu U\nabla^\mu U^\dagger\rangle, \quad (16)$$

$$\mathcal{L}^{(mass)} = \frac{1}{2}BF^2\langle M_\Theta v_2^*U^\dagger + M_\Theta^\dagger v_2U\rangle. \quad (17)$$

Here constants F and B are the same as in (9), while v_1 and v_2 are invariant functions on $\eta^0 + F_0\Theta$. Their normalization is chosen so, that their expansions in the powers of fields start with 1. Due to the parity and charge-conjugation invariance v_1 must be real and even, whereas v_2 may be complex and $v_2^*(x) = v_2(-x)$.¹

Now let us discuss RG properties of lagrangian (13). The crucial question is RG property of the constant F_0 . Strictly speaking, one cannot solve this question until the representation of F_0 is found in terms of QCD parameters. Nevertheless, basing on the common sense, one may assume that F_0 is RG invariant, since its value, according to (12), is related to the normalization of the interpolating field η^0 . Assuming this property (for the strict proof see the next section) and taking into account (7), one gets that $\eta^0 + F_0\Theta$ is RG invariant. The next important observation is that the dependance on A_μ^0 in lagrangian (13) is only realized through $\nabla_\mu\Theta$. As a result and, again, due to (7), RG invariance of \mathcal{L}_{eff} is observed if there is the following renormalization rule, in addition to (11),

$$H_0 = Z^{-1}H_{0R}. \quad (18)$$

Actually, (18) follows from (8) and the fact that parameter H_0 is the decay constant of the axial-vector singlet quark current. The simplest way to verify that is to examine

¹ Notice, sometimes v_2 is defined with the extracted factor $\exp i\lambda^0(\eta^0/F_0 + \Theta)$. Then, the nonet matrix $\Sigma = U \exp(i\lambda^0\eta^0/F_0)$ may be inserted into (17) instead of U , and the external-field combination M instead of M_Θ [10]. However, actually, there are no physical reasons to do this modification since only a few degrees of η^0 are really significant in (17).

this current in the effective theory. In accordance with its natural definition as the variational derivative of the action we have

$$\mathcal{J}_{\mu 5}^0 \equiv \frac{\delta L_{eff}}{\delta A^{0\mu}} = - \frac{\partial \mathcal{L}_{eff}}{\partial (\nabla^\mu \Theta)} = - H_0 \nabla_\mu \eta^0 + H \nabla_\mu \Theta + \dots \quad (19)$$

Here dots mean irrelevant higher-order terms of the expansion in the powers of fields. The straightforward consequence of (19) is $\langle 0 | \mathcal{J}_{\mu 5}^0 | \eta^0 \rangle = -i H_0 p_\mu$, which was to be proved. A more correct proof is based on the analysis of two-point Green function $\delta^2 W / \delta A_\mu^0 \delta A_\nu^0$. When it is considered in QCD, its residue over the pole of η^0 is equal to the matrix element $\langle 0 | \mathcal{J}_{\mu 5}^0 | \eta^0 \rangle$ in square. In the effective theory the direct calculation leads to the squared $H_0 p_\mu$ for this quantity. So, H_0 is really the decay constant of the current $\mathcal{J}_{\mu 5}^0$.

It is interesting to compare (19) with a similar expression for the Noether $U(1)_A$ current. In accordance with (3) and (12) it has the form

$$\mathfrak{J}_{\mu 5}^0 \equiv F_0 \frac{\partial \mathcal{L}_{eff}}{\partial (\partial^\mu \eta^0)} = F_0 \frac{\partial \mathcal{L}_{eff}}{\partial (\nabla^\mu \eta^0)} = F_0 \nabla_\mu \eta^0 + \dots \quad (20)$$

Here F_0 plays a similar role as H_0 in (19) but just in the Noether current. So, F_0 has the meaning of the ‘decay’ constant of the Noether current in the effective theory. However, this meaning is valid no longer in QCD where the Noether $U(1)_A$ current is $\mathcal{J}_{\mu 5}^0 - \mathcal{K}_\mu$ with \mathcal{K}_μ is the pure gauge-field current, $\partial_\mu \mathcal{K}^\mu = \mathcal{Q}$. Really, in QCD the Noether $U(1)_A$ current is conserved. Therefore, in the chiral limit it cannot generate any massive PS state, including η^0 one, since the divergence of the current is zero. On the contrary, in the effective theory the Noether $U(1)_A$ current is not conserved. Therefore, it is able to generate η^0 , which is explicitly exhibited in (20). Let us note, that in fact the $U(1)_A$ symmetry is broken not only in the effective theory but in QCD, too. However, the nature of the breaking is different in both theories. Indeed, in the effective theory the symmetry is explicitly broken due to non-invariance of the lagrangian (with the external fields fixed or switched-off). In QCD the lagrangian is quasi-invariant under global $U(1)_A$, i.e. it transforms on a total divergence (therefore there is the conserved Noether current), and the symmetry is broken due to nonperturbative effects in θ -vacuum [2, 3].

The difference between Noether $U(1)_A$ currents in QCD and in the effective theory may be related to the property that the effective-theory analog of the gluon anomaly is not a divergence. Indeed, in the effective theory the ‘gluon anomaly’ operator is

$$\mathcal{Q} \equiv \frac{\delta L_{eff}}{\delta \Theta} = \left(F_0 \frac{\delta \mathcal{L}_{eff}}{\delta \eta^0} \right) + \lambda^0 \left(S^a \frac{\partial \mathcal{L}_{eff}}{\partial P^a} - P^a \frac{\partial \mathcal{L}_{eff}}{\partial S^a} \right) - \partial_\mu \frac{\partial \mathcal{L}_{eff}}{\partial \nabla_\mu \Theta}. \quad (21)$$

Here the first term in the r.h.s. is neither a divergence and nor a zero when the equation of motion for η^0 is not imposed. So, the same property must be peculiar to \mathcal{Q} , whereas in QCD $\mathcal{Q} \equiv \partial_\mu \mathcal{K}^\mu$. Actually, this difference is natural, because \mathcal{Q} is only able to describe the observable degrees of freedom, and therefore it cannot ‘feel’ the presence of the gauge-variant current \mathcal{K}^μ in QCD. Nevertheless, on the equations of motion both \mathcal{Q} and \mathcal{K} become equivalent. Indeed, from (21) and (19) there follows the anomalous Ward identity for \mathcal{Q} which is the same as that for \mathcal{K} . Owing to (7) the RG property

(8) is satisfied for \mathcal{Q} as well. Notice, with the presence of quarks \mathbf{Q} is completely gauge-invariant operator on the equations of motion (e.g., due to the anomalous Ward identity).

Remember, the operator \mathbf{Q} is able to generate PS glueballs. Besides, it is able to generate the quarkic singlet PS state η^0 , but only in the next-to-leading order in the large- N_c [2, 12]. In the effective theory this property is reproducible, too. Really, with the external fields switched off and the equations of motion taken into account, (21) reads

$$\mathcal{Q} = H_0 M_0^2 \eta^0 + \mu_0 C_0 \eta^0 + \mu_8 C_8 \eta^8 + \dots \quad (22)$$

Here dots stand for multiparticle contributions, the quantities μ_0 and μ_8 are the RG invariant combinations $B(m_u + m_d + m_s)$ and $B(m_u + m_d - 2m_s)$. Parameter C_0 in (22) originates from the power-field expansion of v_2 , $C_8 = \frac{\sqrt{2}}{3}F$. Since $H_0 \sim N_c^{1/2}$ and $M_0^2 \sim N_c^{-1}$ at large N_c , then it follows from (22) that in the limit of the massless quarks $\mathcal{Q} \sim N_c^{-1/2}$, whereas the correct behavior is $\mathcal{Q} \sim N_c^0$. To reproduce this correct behavior one has to take into consideration another singlet state, η^G , which is gluonic by its origin. The corresponding generalization of the lagrangian is obvious: one should allow for the η^G -dependence in $v_{1,2}$ and the necessary terms in $\mathcal{L}^{(0)}$. The latter one now reads

$$\begin{aligned} \mathcal{L}^{(0)} = & \frac{1}{2} \nabla_\mu \eta^0 \nabla^\mu \eta^0 - \frac{1}{2} M_0^2 (\eta^0 + F_0 \Theta)^2 \\ & + \frac{1}{2} \partial_\mu \eta^G \partial^\mu \eta^G - \frac{1}{2} M_G^2 (\eta^G)^2 - q (\eta^0 + F_0 \Theta) \eta^G \\ & + H_0 \nabla_\mu \eta^0 \nabla^\mu \Theta + H_G \partial_\mu \eta^G \nabla^\mu \Theta + \frac{1}{2} H \nabla_\mu \Theta \nabla^\mu \Theta + \mathcal{L}_{int}^{(0)} \quad . \end{aligned} \quad (23)$$

Here q is a new parameter that describes the mixing between η^0 and η^G , M_G is a mass parameter for η^G , H_G is another decay constant of the current $J_{\mu 5}^0$. The gluonic nature of η^G is developed in the specific large- N_c behavior of its parameters. Namely, since η^G is a gluonic state, then $H_G \sim N_c^0$ and $M_G^2 \sim N_c^0$. Besides, since η^0 is a quarkic state, then $q \sim N_c^{-1/2}$. (See [10] and [12] for the way to show that.) So, now

$$\begin{aligned} \mathcal{Q} = & (H_G M_G^2 + H_0 q) \eta^G + (H_0 M_0^2 + H_G q) \eta^0 \\ & + \mu_0 C_0 \eta^0 + \mu_8 C_8 \eta^8 + \dots \end{aligned} \quad (24)$$

has the correct behavior at large N_c , which is caused by the presence of η^G .

It is important to note that in (23) we did not introduce the kinetic-mixing term $\nabla_\mu \eta^0 \partial^\mu \eta^G$ because we considered η^0 and η^G to be independent canonical variables. The latter property follows from the condition that η^0 and η^G must describe quite different degrees of freedom. (Namely, the quarkic and gluonic ones, which we assume to exist on the projection to interpolating fields in QCD. This is quite a general assumption and we do not consider it as a model-dependent one. Notice, an equivalent assumption reads that there exist glueballs and meson quarkic states in QCD.)

The RG properties of the parameters of η^G in lagrangian (23) may be established analogously to those of η^0 . In this way, q and M_G must be RG invariant, like F_0 does, in order to provide the lagrangian with RG invariance. Since H_G is the decay constant of the axial-vector singlet quark current, its renormalization rule must coincide with that of H_0 . An important consequence of these RG properties is the scale-dependent mixing

of the quarkic and gluonic contributions to the gluon anomaly operator. Looking at (24) we obtain this property due to RG non-invariance of the first two terms and invariance of the last two terms. In QCD a similar RG behavior of the gluon anomaly was established in [14] and discussed in detail in [7]. This behavior means that in QCD with quarks the gluon anomaly operator has no pure gluonic nature, but it rather has a mixed nature. Actually, the mixed nature is peculiar to other QCD composite operators which mix under RG. Especially it becomes clear when the operators are considered on the equations of motion, since even the fundamental fields of quarks and gluons carry the mixed degrees of freedom on the equations of motion.

So, the composite operators appear to be not quite suitable variables for description of singlet states. The preferable variables appear to be the quarkic and gluonic interpolating fields because they are RG invariant and available for a direct description of the quarkic and gluonic degrees of freedom of the observable states. In the framework of QCD these interpolating fields may be introduced in rather indirect manner, basing on the composite operators as the initial objects [7]. In the framework of the effective theory these interpolating fields are introduced as the fundamental objects. This peculiarity shows a certain advantage in describing singlet states in the framework of the effective theory.

Now, so long as the lowest quarkic and gluonic states have been introduced, one can make the next step and introduce other singlet states. Assuming that each new singlet state presents its own unique degree of freedom, one has to assume that its interpolating field must be independent canonical variable. In what follows we will reserve the symbol η^κ for any extra singlet PS state if it does not coincide with η^0 and we are not interested in its nature. Special attention will be paid to the gluonic ground-state η^G and an excitation state over η^0 , which will be designated by the symbol $\tilde{\eta}^0$. The difference between η^G and $\tilde{\eta}^0$ may be detected in their large- N_c behavior. For instance, parameter \tilde{q} , which describes the mixing between η^0 and $\tilde{\eta}^0$, must behave as $\tilde{q} \sim N_c^{-1}$ whereas $q \sim N_c^{-1/2}$. In principle, in this way one may unambiguously distinguish between the states η^G and $\tilde{\eta}^0$. However, the large- N_c approach alone will hardly be useful to obtain any significant phenomenological result. A more promising way seems to be in searching for more strong constraints for singlet-state contributions to the lagrangian.

4 Singlet-state constraints

Let us return to the problem of the physical significance of F_0 . We have seen that F_0 does not contribute to the axial-vector singlet quark current and to the gluon anomaly operator. Therefore F_0 cannot be expressed in terms of QCD Green functions with corresponding legs. Let us examine now the pseudoscalar quark current. First, one must define the power-field expansion of the invariant function v_2 ,

$$v_2 = 1 + i\lambda^0 \left\{ b_0(\eta^0/F_0 + \Theta) + \sum_{\kappa} b_{\kappa}\eta^{\kappa}/F \right\} + \dots \quad (25)$$

Here $\lambda^0 = \sqrt{2/3}$ is the numerical factor, b_0 and b_{κ} are the parameters of the linear term of the expansion, dots stand for the higher-order terms of the expansion. In view of (25)

and (17) the PS quark current in the effective theory is

$$\mathcal{J}_5^0 \equiv -\delta L_{eff}/\delta P^0 = -BF^2 \left(b_0 \eta^0/F_0 + \sum_{\kappa} b_{\kappa} \eta^{\kappa}/F \right) + \dots \quad (26)$$

Here in the r.h.s the sources are switched off. Now let us take into account the equality $\langle 0|\mathcal{J}_5^0|\eta^0 \rangle = \langle 0|J_5^0|\eta^0 \rangle$ which follows from the equality of the residues over the η^0 -pole in Green function $\delta^2 W/\delta P^0 \delta P^0$ in the effective theory and in QCD. Then, owing to $BF^2 = -\langle \bar{u}u \rangle_0$, where $\langle \bar{u}u \rangle_0$ is the chiral quark condensat ($\langle \bar{u}u \rangle_0 = \langle \bar{d}d \rangle_0 = \langle \bar{s}s \rangle_0$), one may obtain from (26) the relation

$$\frac{F_0}{b_0} = \frac{\langle \bar{u}u \rangle_0}{\langle 0|J_5^0|\eta^0 \rangle}. \quad (27)$$

Owing to (27) the physical significance of the ratio F_0/b_0 is clear: it presents the coefficient which should be extracted with the inverse quark condensat from the QCD composite operator J_5^0 in order to obtain the canonically normalized interpolating η^0 field on the mass shell of η^0 . Note, a similar expression for this coefficient was obtained in [5] where also the RG invariance property of this coefficient was discussed.

However, the physical significance of the parameter F_0 proper still remains unclear. In this connection, let us also examine the $U(1)_A$ transformation property of the current $J_5^0 = i\bar{q}\gamma_5\lambda^0/2q$. To this end let perform rotation $q \rightarrow \exp(-i\gamma_5\omega_5^0\lambda^0/2)q$, $\bar{q} \rightarrow \bar{q}\exp(-i\gamma_5\omega_5^0\lambda^0/2)$. Then obtain with infinitesimal ω_5^0

$$J_5^0 \rightarrow J_5^0 + J^0\lambda^0\omega_5^0. \quad (28)$$

Here $J^0 = \bar{q}\lambda^0/2q$. Now let us take into consideration the expansions in the powers of interpolating fields of the currents J_5^0 and J^0 :

$$J_5^0 = \langle 0|J_5^0|\eta^0 \rangle \eta^0 + \sum_{\kappa} \langle 0|J_5^0|\eta^{\kappa} \rangle \eta^{\kappa} + \dots \quad (29)$$

$$J^0 = (\lambda^0)^{-1} \langle \bar{u}u \rangle_0 + \dots \quad (30)$$

Here in (29) η^0 and η^{κ} are canonically normalized interpolating fields for $|\eta^0 \rangle$ and $|\eta^{\kappa} \rangle$, dots stand for multiparticle contributions. In (30) the first term in the r.h.s. is the v.e.v. of the current J^0 , dots stand for the irrelevant scalar-particle contributions and multiparticle contributions. Substitute (29) and (30) into (28). Then, assuming the chiral-invariance of η^{κ} , we obtain that η^0 should transform on a shift:

$$\eta^0 \rightarrow \eta^0 + \frac{\langle \bar{u}u \rangle_0}{\langle 0|J_5^0|\eta^0 \rangle} \omega_5^0. \quad (31)$$

Note, this η^0 has been defined directly in QCD. So, if one identifies this η^0 with that in the effective theory, then one confirms condition (12). Moreover, its origin becomes clear: it is the consequence just of the nonzero v.e.v. of the current J^0 which is the chiral partner of J_5^0 . Comparing (12) with (31), one can also deduce the QCD representation of F_0 . It turns out to be exactly the r.h.s. of (27). So, since the normalization of η^0 is the

same everywhere (the canonical one), there is consistency condition on the parameter b_0 ,

$$b_0 = 1. \quad (32)$$

In view of (32) and (27) the physical significance of F_0 is found to be the same which was before for the ratio F_0/b_0 . Simultaneously, both in QCD and in the effective theory parameter F_0 governs the $U(1)_A$ transformation property of the interpolating η^0 field.

Actually, consistency condition (32) is the universal one since it is valid not matter what number of extra singlet states has been involved/integrated out. So, condition (32) is able to constrain the contributions of other singlet states to the chiral effective lagrangian. Indeed, if any η^κ contributes linearly to expansion (25), i.e. if $b_\kappa \neq 0$, then it should be $q_\kappa = 0$, that is η^κ does not mix with η^0 in $\mathcal{L}^{(0)}$. (Otherwise, after η^κ is integrated out there will appear an extra dependence on η^0 in (25) which is caused by the former mixing, and this extra dependence will break condition (32).) If, on the contrary, $b_\kappa = 0$, then it well may be $q_\kappa \neq 0$. Moreover, in accordance with the Weinberg ‘theorem’ [8] if it is allowed $q_\kappa \neq 0$, then it should be $q_\kappa \neq 0$.

So, we have obtained a strict result that η^κ cannot contribute simultaneously to the linear term of the expansion of v_2 and to the mixing $\eta^0\eta^\kappa$ -term in lagrangian $\mathcal{L}^{(0)}$. Therefore, there are two quite different ways to involve an extra singlet state to the effective theory. This fact inspires an idea that each way corresponds to some specific kind of the singlet state. In the reality that indeed takes place. To show this let us consider the limit of the massless quarks, when the octet states become the Goldstone bosons but η^0 and η^κ remain massive states. Then the condition $q_\kappa = 0$ means that η^κ cannot be converted into the ground-state η^0 without the emission of some number of Goldstone bosons. However, this behavior is peculiar exactly to excited states, since when there is no mass (energy) gap massless strong-interacting particles *should necessarily be emitted* in course of any transformation of an excited state. (One may consider this property as an independent definition of excited states in the framework of the effective theory.) So, the condition $q_\kappa = 0$ may be considered as the necessary condition for excited states. Another case, when η^κ does not contribute to the linear term in v_2 ($b_\kappa = 0$), is quite natural for glueballs. Indeed, since gluons do not distinguish the quark flavors, a pure glueball *cannot contribute directly* (through the vertex) to any process which breaks the flavor symmetry. Therefore, the glueball interpolating field cannot appear in lagrangian $\mathcal{L}^{(mass)}$ which explicitly breaks the flavor symmetry.² The latter property should be regarded as the necessary condition for glueballs.

The results of the discussion, which concern the states η^G and $\tilde{\eta}^0$, are summarized in the Table,

² It is interesting to note, that, consequently, glueballs cannot contribute to scalar and pseudoscalar quark currents \mathcal{J}^0 and \mathcal{J}_5^0 (J^0 and J_5^0). Besides, the pure PS glueball η^G does not contribute to \mathcal{Q} (Q) through quark-mass-dependent terms (see Eq. (24); cf. [7]).

The kind of the extra singlet PS state (η^κ)	contributes into v_2 in $\mathcal{L}^{(mass)}$	mixes with η^0 in $\mathcal{L}^{(0)}$
ground-state glueball (η^G) excited state ($\tilde{\eta}^0$)	no ($b_G = 0$) yes ($\tilde{b}_0 \neq 0$)	yes ($q \neq 0$) no ($\tilde{q} = 0$)

In addition to the Table, it should be noted that by the above reasons the mixing between η^G and $\tilde{\eta}^0$ is suppressed in $\mathcal{L}^{(0)}$. Besides, any excitation of the PS glueball ($\tilde{\eta}^G$) does not contribute both to the linear term of v_2 ($\tilde{b}_G = 0$) and to the mixing $\tilde{\eta}^G \eta^0$ -term in $\mathcal{L}^{(0)}$ ($\tilde{q}_G = 0$). A analogous analysis may be extended to any other singlet state.

5 Mixing

The results of the previous section are most important for investigation of the spectrum of singlet states. Let us discuss briefly this question making the accent on the mixing phenomenon. The simplest case is when only the ground-states η^0 and η^G are involved as singlet states. Then, neglecting the isotopic symmetry breaking ($m_u = m_d \neq m_s$), one has three iso-singlet mixing states: η^8 , η^0 , and η^G . In virtue of the above Table and (13), (17), (23), the squared mass matrix in the basis of these states is

$$\mathcal{M}^2 = \begin{pmatrix} d_8 & ra & 0 \\ & M_0^2 + \beta_0 d_0 & q \\ \text{symm.} & & M_G^2 \end{pmatrix}. \quad (33)$$

Here

$$d_8 = \frac{1}{3}(4M_K^2 - M_\pi^2), \quad d_0 = \frac{1}{3}(2M_K^2 + M_\pi^2), \quad a = \frac{2\sqrt{2}}{3}(M_\pi^2 - M_K^2),$$

M_π and M_K are pion and kaon masses, $r = F/F_0$, and β_0 is the parameter of the quadratic $(\eta^0)^2$ -term in the expansion of v_2 . Matrix \mathcal{M}^2 may be diagonalized by the orthogonal rotation matrix

$$\mathcal{O} = [\mathcal{O}]_n^j = \begin{pmatrix} c_2 c_3 & s_2 & c_2 s_3 \\ s_1 s_3 - c_1 s_2 c_3 & c_1 c_2 & -s_1 c_3 - c_1 s_2 s_3 \\ -c_1 s_3 - s_1 s_2 c_3 & s_1 c_2 & c_1 c_3 - s_1 s_2 s_3 \end{pmatrix}. \quad (34)$$

Here $c_i = \cos \theta_i$, $s_i = \sin \theta_i$; $\theta_1 = \theta_{0-G}$, $\theta_2 = \theta_{8-0}$, $\theta_3 = \theta_{8-G}$. The row index j and the column index n run the values $j = 8, 0, G$ and $n = \eta, \eta', \eta^{G_l}$.

One can obtain the following relations between the angles θ_i and the parameters of the matrix \mathcal{M}^2 ,

$$\tan \theta_1 = \mathcal{O}_{\eta'}^G / \mathcal{O}_{\eta'}^0 = -\frac{q}{M_G^2 - M_0^2 - \beta_0 d_0}, \quad (35)$$

$$\tan \theta_2 = c_1 \cdot \mathcal{O}_{\eta'}^8 / \mathcal{O}_{\eta'}^0 = \frac{ac_1}{M_0^2 + \beta_0 d_0 - d_8}, \quad (36)$$

$$\tan \theta_3 = -\frac{s_1 + c_1 s_2 t_3}{c_2} \cdot \mathcal{O}_{\eta^{G_l}}^8 / \mathcal{O}_{\eta^{G_l}}^0 = -\frac{as_1}{M_G^2}. \quad (37)$$

Here in each formula the second equality displays the leading term of the combined chiral and large- N_c expansion. Remember, $M_{\pi,K}^2 = O(p^2, 1)$, $M_0^2 = O(1, N_c^{-1})$, $M_G^2 = O(1, 1)$, $q = O(1, N_c^{-1/2})$. Consequently, $M_\eta^2 = O(p^2, 1)$, $M_{\eta'}^2 = O(p^2, N_c^{-1})$, $M_{\eta^{GI}}^2 = O(1, 1)$.

In what follows, we will use a common parameter ε of the combined expansion. We choose it to be of the order $O(p)$ in the sense of the chiral expansion and of the order of some negative power of N_c in the sense of the large- N_c expansion,

$$\varepsilon = O(p) = O(N_c^{-\alpha}). \quad (38)$$

Actually, one should consider $\alpha > 1/2$, since otherwise η^0 could not be a heavy state as compared with pions and kaons. The real value of α may be estimated like as follows. Let, in accordance with (38), $M_\eta^2 \sim \varepsilon^2$, $M_{\eta'}^2 \sim \varepsilon^{1/\alpha}$, $M_{\eta^{GI}}^2 \sim 1$. Then one can obtain $\alpha \simeq \frac{1}{2} \ln(M_{\eta^{GI}}^2/M_\eta^2) / \ln(M_{\eta^{GI}}^2/M_{\eta'}^2)$. As a consequence, $\alpha = 0.9$ – 1.1 in wide mass region $M_{\eta^{GI}} = (1.5$ – $1.9)$ GeV. So, one may put approximately $\alpha \simeq 1$. With this value of α from (35)–(37) we have the hierarchy of the angles:

$$\theta_1 = \theta_{0-G} \sim \varepsilon^{1/2}, \quad \theta_2 = \theta_{8-0} \sim \varepsilon, \quad \theta_3 = \theta_{8-G} \sim \varepsilon^{5/2}. \quad (39)$$

Note, the mixing $\eta^8 - \eta^G$ is the smallest one because it arises only due to intermediate mixing with η^0 .

Now let us consider the most interesting case when η^G and $\tilde{\eta}^0$ are both involved together with η^0 . The squared mass matrix in the basis $\eta^8 - \eta^0 - \eta^G - \tilde{\eta}^0$ has the form³

$$\mathbf{M}^2 = \begin{pmatrix} & & & \vdots & \tilde{b}_0 a \\ & \mathcal{M}^2 & & \vdots & \tilde{\beta}_0 d_0 \\ & & & \vdots & 0 \\ \cdots & \cdots & \cdots & \vdots & \cdots \\ \tilde{b}_0 a & \tilde{\beta}_0 d_0 & 0 & \vdots & M_{\eta^0}^2 \end{pmatrix}. \quad (40)$$

Here \mathcal{M}^2 is given by (33). The zeros in (40) reflect the absence of the direct mixing between η^G and $\tilde{\eta}^0$. Parameter $\tilde{\beta}_0$ describes the mixing $\eta^0 - \tilde{\eta}^0$ caused by lagrangian $\mathcal{L}^{(mass)}$ (the $\eta^0 \tilde{\eta}^0$ -term in the expansion of v_2).

The diagonalization of the matrix \mathbf{M}^2 may be performed in two steps. First, one may diagonalize the block \mathcal{M}^2 :

$$\begin{aligned} \mathbf{M}^2 \rightarrow \mathbf{M}_\mathcal{O}^2 &= \begin{pmatrix} \mathcal{O} & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & 1 \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{M}^2 & \vdots & \delta \mathcal{M}^2 \\ (\delta \mathcal{M}^2)^T & \vdots & M_{\eta^0}^2 \end{pmatrix} \begin{pmatrix} \mathcal{O} & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & 1 \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{M}_\mathcal{O}^2 & \vdots & \delta \mathcal{M}_\mathcal{O}^2 \\ (\delta \mathcal{M}_\mathcal{O}^2)^T & \vdots & M_{\eta^0}^2 \end{pmatrix}. \end{aligned} \quad (41)$$

³ Strictly speaking, involving $\tilde{\eta}^0$, one must also take into account $\tilde{\eta}^8$, where $\tilde{\eta}^8$ is the eighth member of the nonet of excited states. (A general way to include the octet heavy states is discussed in [11].) Then, $\tilde{\eta}^0$ and $\tilde{\eta}^8$ must mix in lagrangian $\mathcal{L}^{(mass)}$ to produce two final states. However, one of these final states will decouple practically from the further mixing with η^8 , η^0 , η^G , since $m_{u,d} \ll m_s$. As a result, only the other state is really significant. Since its contribution to the mixing is quite similar to that of $\tilde{\eta}^0$, considered without $\tilde{\eta}^8$, we will neglect for simplicity the effect of the presence of $\tilde{\eta}^8$.

Here $\mathcal{M}_\mathcal{O}^2 = \text{diag}(M_1^2, M_2^2, M_3^2)$, and $\delta\mathcal{M}^2$ is the upright block in (40),

$$\delta\mathcal{M}_\mathcal{O}^2 \equiv \mathcal{O}^T \delta\mathcal{M}^2 = \sqrt{4/3} M_K^2 \mathcal{Y}. \quad (42)$$

In (42) $\mathcal{Y} = \text{column}(Y_1, Y_2, Y_3)$, $Y_n = -\sqrt{2/3} \mathcal{O}_n^8 \tilde{b}_0 + \sqrt{1/3} \mathcal{O}_n^0 \tilde{\beta}_0$, and we neglected the u and d quark contributions as compared with the s -ones (approximation $M_\pi^2 \ll M_K^2$). Matrix $\mathbf{M}_\mathcal{O}^2$ may be diagonalized by the next transformation

$$\begin{aligned} \mathbf{M}_\mathcal{O}^2 \rightarrow \mathbf{M}_{\Omega_\sigma}^2 &= \begin{pmatrix} 1 & \vdots & -\sigma \\ \cdots & \ddots & \cdots \\ \sigma^T & \vdots & 1 \end{pmatrix} \begin{pmatrix} \mathcal{M}_\mathcal{O}^2 & \vdots & \delta\mathcal{M}_\mathcal{O}^2 \\ \cdots & \ddots & \cdots \\ (\delta\mathcal{M}_\mathcal{O}^2)^T & \vdots & M_{\eta^0}^2 \end{pmatrix} \begin{pmatrix} 1 & \vdots & \sigma \\ \cdots & \ddots & \cdots \\ -\sigma^T & \vdots & 1 \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{M}_\mathcal{O}^2 - \delta\mathcal{M}_\mathcal{O}^2 \sigma^T - \sigma (\delta\mathcal{M}_\mathcal{O}^2)^T + \sigma M_{\eta^0}^2 \sigma^T & \vdots & \mathcal{M}_\mathcal{O}^2 \sigma - \sigma M_{\eta^0}^2 + \delta\mathcal{M}_\mathcal{O}^2 - \sigma (\delta\mathcal{M}_\mathcal{O}^2)^T \sigma \\ \cdots & \ddots & \cdots \\ \sigma^T \mathcal{M}_\mathcal{O}^2 - M_{\eta^0}^2 \sigma^T + (\delta\mathcal{M}_\mathcal{O}^2)^T - \sigma^T \delta\mathcal{M}_\mathcal{O}^2 \sigma^T & \vdots & M_{\eta^0}^2 + (\delta\mathcal{M}_\mathcal{O}^2)^T + \sigma^T \delta\mathcal{M}_\mathcal{O}^2 + \sigma^T \mathcal{M}_\mathcal{O}^2 \sigma \end{pmatrix} \end{aligned} \quad (43)$$

Here the transformation matrix approaches the orthogonal one if $\sigma = \mathcal{O}(p^2)$ (we do not consider temporarily the large- N_c expansion). Assuming this property, we obtain that matrix $\mathbf{M}_{\Omega_\sigma}^2$ is diagonal at order p^2 if

$$\mathcal{M}_\mathcal{O}^2 \sigma - \sigma M_{\eta^0}^2 + \delta\mathcal{M}_\mathcal{O}^2 = 0. \quad (44)$$

From (44) and (42) it follows that

$$\sigma_n = \sqrt{4/3} \frac{M_K^2}{M_{\eta^0}^2 - M_n^2} \mathcal{Y}_n. \quad (45)$$

In view of $M_K^2 = \mathcal{O}(p^2)$, $M_{\eta^0}^2 = \mathcal{O}(1)$, one can deduce from (45) that σ is really of order p^2 . So, up to and including order p^2 of the chiral expansion the total diagonalizing matrix is

$$\Omega_\sigma = \begin{pmatrix} \mathcal{O} & \vdots & \mathcal{O}\sigma \\ \cdots & \ddots & \cdots \\ -\sigma^T & \vdots & 1 \end{pmatrix}, \quad (46)$$

where $\mathbf{M}_{\Omega_\sigma}^2 = \Omega_\sigma^{-1} \mathbf{M}^2 \Omega_\sigma$. In (46) each σ_n may be understood as the angle describing the mixing between $\tilde{\eta}^0$ and η^8, η^0, η^G . Owing to (34), (39) and $\tilde{b}_0, \tilde{\beta}_0 \sim N_c^0$ we have

$$\theta_{8-\tilde{0}} \simeq \sigma_1 \sim \varepsilon^2, \quad \theta_{0-\tilde{0}} \simeq \sigma_2 \sim \varepsilon^2, \quad \theta_{G-\tilde{0}} \simeq \sigma_3 \sim \varepsilon^{5/2}. \quad (47)$$

So, θ_{0-G} and θ_{0-8} remain to be the main mixing angles with the behavior of $\varepsilon^{1/2}$ and ε , respectively. The next angles turn out to be $\theta_{8-\tilde{0}}$ and $\theta_{0-\tilde{0}}$, both of order ε^2 . The angles $\theta_{G-\tilde{0}}$ and θ_{8-G} are the smallest ones because they arise non-directly only, through the intermediate mixing with η^0 (namely, through the mixings $\eta^0 - \tilde{\eta}^0$ in $\mathcal{L}^{(mass)}$ and $\eta^0 - \eta^G$ in $\mathcal{L}^{(0)}$).

It should be noticed, that there is one dangerous case in the above general picture. It is when the denominator in (45) is small. Such situation may take place when n has the meaning of η^{G^i} , since only in this case $M_n^2 = \mathcal{O}(1)$, as well as $M_{\eta^0}^2$ does. Then, due to the ‘play of numbers’ the difference $M_{\eta^0}^2 - M_{\eta^{G^i}}^2$ may take any value, including one which is close to the value of the numerator in (45). If this situation does take place, then one has to reconsider the above analysis, rejecting the approximation scheme (43)–(47) and making instead numerical estimates.

6 Radiative decays

According to the widely spread opinion, radiative decays are the best tool for the phenomenological investigation of the PS meson mixing. Usually, the well-known PCAC formulae are drawn for this purpose. However, as we pointed out in Introduction, in the singlet channel the usual PCAC formula was valid no longer. The corrected PCAC formula was proposed in [5]. It involves a new ‘decay’ constant instead of the usual axial-vector-current one and an additional proper vertex. (Actually, [5] discussed the process $\eta' \rightarrow \gamma\gamma$ with the interpolating η' field defined on the base of the properly normalized current $J_5^0 = i\bar{q}\gamma_5\lambda^0/2q$. So, the η' of [5] and our η^0 are the same objects. Notice, [5] did not consider, however, the mixings of η' .)

It would be worth investigating the decay $\eta^0 \rightarrow \gamma\gamma$ in the approach of the chiral effective lagrangian. In fact, the first result of [5], which concerns the appearance of a new ‘decay’ constant in the correct formula, is reproduced trivially. Indeed, when $U(1)_A$ symmetry is taken into consideration, then the p^4 -order WZW term, which is responsible for two-photons decays, must involve the nonet-field matrix $\Sigma = U \exp(i\lambda^0\eta^0/F_0)$ instead of the usual octet matrix U . Since Σ involves the singlet interpolating field η^0 divided by F_0 , the appearance of a new ‘decay’ constant is obvious. The second result, which concerns the presence of an additional proper vertex in the correct formula, may be reproduced, too. Its origin in our approach is related with the presence in the chiral effective lagrangian of an additional chiral-invariant term of order p^4 , which contains the totally antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma}$, and which, notwithstanding, is parity-even,

$$\mathcal{L}_{eff} = \dots + \mathcal{L}_{WZW} + v_3 \epsilon_{\mu\nu\rho\sigma} < F_L^{\mu\nu} F_L^{\rho\sigma} + F_R^{\mu\nu} F_R^{\rho\sigma} >. \quad (48)$$

In (48) dots mean the usual chiral-invariant lagrangian, which is irrelevant to the discussion, \mathcal{L}_{WZW} is the WZW lagrangian, the last term is the very additional one. The quantity v_3 is a chiral-invariant function with positive charge conjugation and negative parity. So, the parity and the charge conjugation of the additional term are correct. Moreover, the additional term is RG invariant as well, since it does not coincide with the external-field counterterm discussed in [14] (see, also, Section 2) and v_3 depends on RG invariant variables.

It is easy to detect the contribution from the additional term to the process $\eta^0 \rightarrow \gamma\gamma$. Indeed, let the power expansion of v_3 be

$$v_3 = g_0(\eta^0 + F_0\Theta) + \sum_{\kappa} g_{\kappa}\eta^{\kappa} + \dots \quad (49)$$

with g_0 and g_{κ} the constants. Substituting (49) into (48) and extracting only terms with two photons, one may conclude that g_0 is really the proper vertex that contributes to $\eta^0 \rightarrow \gamma\gamma$. However, unlike [5], we have not any reason to interpret it as the “coupling of the glue component of η' to photons”. Indeed, η^0 describes quarkic degrees of freedom (on the projection to interpolating fields) whereas gluonic ones are described by η^G . So, the coupling of the “glue” to photons might rather be peculiar to term $g_G\eta^G$ but not to $g_0\eta^0$. Moreover, in our opinion the coupling g_G must be equal to zero, since the pure “glue” cannot interact directly with photons.⁴ Therefore the term $g_G\eta^G$ must

⁴ Nevertheless, an indirect interaction of gluons to photons is allowed. This property is represented

be suppressed in (49). On the base of this reason we can think, also, that the “glue” proper vertex of [5] in the reality is equal to zero, as well. (Let us emphasize, that the proper vertex of [5] was introduced in rather indirect manner; the necessity of its presence was not supported by any other observables, and no arguments were presented why it was nonzero.) Nevertheless, there is the η^0 -proper vertex. One may detect it in the framework of the approach of [5] if one rejects the approximation $k^2 = 0$ in equation (4.2) of [5] and instead considers the mass-shell condition $k^2 = M_{\eta'}^2$ with $M_{\eta'}^2 \neq 0$. Note, although the condition $k^2 = 0$ is usual in PCAC, it is not an adequate approximation in case of non-Goldstone states.

It is easy to show that $g_0 F_0 \sim N_c^0$ at large N_c . Therefore $g_0 \sim N_c^{-1/2}$. So, taking into account the factor $N_c^{1/2}$ in the WZW term, one may conclude that g_0 contributes to the amplitude of $\eta^0 \rightarrow \gamma\gamma$ in the next-to-leading order in the large- N_c . Nevertheless, the contribution of the proper vertex is not negligible in the phenomenological sense. On the contrary, when one considers the real decay $\eta' \rightarrow \gamma\gamma$, then the proper vertex will contribute in the same order of the combined chiral and large- N_c expansion in which the mixing $\eta^0 - \eta^8$ contributes. Really, taking into account the WZW factor $N_c^{1/2}$ and disregarding the overall factor, the amplitude of $\eta' \rightarrow \gamma\gamma$ may be represented as

$$\begin{aligned} A_{\eta' \rightarrow \gamma\gamma} &\propto \Omega_{\eta'}^8 + 2\sqrt{2}r\Omega_{\eta'}^0 + N_c^{-1/2}g_0\Omega_{\eta'}^0 \\ &= s_2 + 2\sqrt{2}rc_1c_2 + N_c^{-1/2}g_0c_1c_2. \end{aligned} \quad (50)$$

Here we have used (34) and only take into account the contributions of η^8 and η^0 (other ones are really negligible). Basing on the results of the previous section one can see that the second term in the r.h.s is the leading one. It behaves as $O(1)$ and describes the η^0 -contribution that arises from the WZW term. The first term describes the WZW η^8 -contribution. The third term is caused by the additional term in (48). Since these latter two terms behave as $O(p^2N_c)$ and $O(N_c^{-1})$, respectively, they both belong to one and the same (next-to-leading) order $O(\varepsilon)$ of the combined chiral and large- N_c expansion.

The above analysis shows that when studying the decay $\eta' \rightarrow \gamma\gamma$ with the $\eta - \eta'$ mixing is neglected, one must also neglect the proper vertex g_0 . However, if the $\eta - \eta'$ mixing is taken into account, then one must take into account the $\eta^0 - \eta^8$ mixing, the proper vertex g_0 , and the effect $F_0 \neq F$ ($r = 1 + O(N_c^{-1})$), since all these effects contribute to the amplitude of the decay in one and the same order of the combined chiral and large- N_c expansion. On the contrary, the PCAC formula for $\eta \rightarrow \gamma\gamma$ works well with the $\eta - \eta'$ mixing taken into account, since the additional term in (48) and the effect $F_0 \neq F$ contribute to the amplitude of the decay $\eta \rightarrow \gamma\gamma$ starting at order $O(\varepsilon^2)$, i.e. in the next-next-to-leading order whereas the contribution of the $\eta^8 - \eta^0$ mixing is of order $O(\varepsilon)$.

in (49) by the term $g_0 F_0 \Theta$ which describes the QCD Green function (not the vertex!) of the gluon anomaly operator and two photons. Notice, the coupling constant of this term coincides with the vertex g_0 due to $U(1)_A$ invariance.

7 Summary and discussion

The present paper proposes a model-independent way to constrain the contributions of the singlet PS states to the chiral effective lagrangian with an accounting their nature. This allows one to independently identify the singlet states in the framework of the approach, and it opens the way to systematic investigation of the low-lying singlet resonances in QCD. Moreover, it becomes possible to interpret more precisely the results of some previous investigations. For instance, one should conclude that the PS ‘glueball’ of [15] is not really a glueball but rather an excitation of the singlet quarkic state, because in [15] it was included into lagrangian $\mathcal{L}^{(mass)}$ which involved the current quark masses and, therefore, explicitly broke the flavor symmetry.

A special attention is paid in the present paper to construct a correct generalization of the chiral effective lagrangian which would involve singlet interpolating fields and satisfy not only the chiral symmetry but the QCD-inspired RG symmetry, too. Correct account of RG symmetry allows us to introduce the gluonic and singlet quarkic interpolating fields to be RG invariant objects which separately describe the gluonic and singlet quarkic degrees of freedom in the effective theory. Owing to dynamical reasons these interpolating fields may mix in the lagrangian. However, this mixing remains RG invariant in spite of the fact that the relevant composite operators in QCD mix under RG. This property shows a certain advantage in describing singlet states in the framework of the effective theory.

Besides the above results, which have rather general significance, the present paper also proposes some particular results. Thus the mixing among the iso-singlet PS states is investigated and the hierarchy of the mixing angles is obtained which is defined by the combined chiral and large- N_c expansion. The largest mixing angle is seen to be between the singlet quarkic lowest state η^0 and the gluonic ground-state η^G . Then, in order of decreasing significance, follow the $\eta^8 - \eta^0$ mixing, and the $\eta^0 - \tilde{\eta}^0$, $\eta^8 - \tilde{\eta}^0$ mixings, where $\tilde{\eta}^0$ is the excitation of η^0 . The mixing angles $\eta^G - \eta^8$, $\eta^G - \tilde{\eta}^0$ turn out to be the smallest ones and negligible in the approximation up to and including $O(\varepsilon^2)$ where ε is the parameter of the expansion.

Another important application concerns the radiative decays $\eta \rightarrow \gamma\gamma$ and $\eta' \rightarrow \gamma\gamma$. We reproduce the modern PCAC results [5], i.e. we show that the correct formula for $\eta' \rightarrow \gamma\gamma$ must involve a special ‘decay’ constant instead of the usual axial-vector-current decay constant, and an additional proper vertex. However, the nature of the proper vertex is found different in our approach as compared with that of [5]. Besides, we show that the proper vertex contributes to the amplitude of the decay $\eta' \rightarrow \gamma\gamma$ in the same order of the combined chiral and large- N_c expansion in which the η^8 -state contributes due to the $\eta - \eta'$ mixing. Therefore, the effect of the proper vertex must be considered together with the mixing. On the contrary, the well-known PCAC formula for the decay $\eta \rightarrow \gamma\gamma$ works well without any modifications, even when the mixing $\eta - \eta'$ is taken into consideration. So, this decay remains a good tool for the study of the mixing. (Nevertheless, the proper vertex contributes to the amplitude of $\eta \rightarrow \gamma\gamma$ in the next-to-leading order where the octet decay constant F is split, $F_8 \neq F_\pi$. So, the p^4 -order corrections to $\eta \rightarrow \gamma\gamma$ must be taken into account together with the proper vertex.)

On the whole, the results of this paper establish a formal framework which to perform

quantitative estimates. Such work should take into account the real spectrum of the observed mesons and the data on their decays. One might obtain the quantitative description of the mixing as the output result, which is necessary for interpretation of the nature of the observed mesons. Of course, an extension of the analysis to other problems and to other channels would be possible. These questions will be addressed in forthcoming papers.

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